

Closed subgroups of the polynomial automorphism group containing the affine subgroup.

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Abstract

We prove that, in characteristic zero, closed subgroups of the polynomial automorphisms group containing the affine group contain the whole tame group.

1 Introduction

Throughout, \mathbb{K} denotes an algebraically closed field of characteristic $\text{char}(\mathbb{K})$ and $n \in \mathbb{N}_+$ denotes a positive integer.

We denote by $\mathcal{E} = \text{End}_n(\mathbb{K})$ the set $(\mathbb{K}[x_1, \dots, x_n])^n$ endowed with the following law $\sigma\tau = (f_1(g_1, \dots, g_n), \dots, f_n(g_1, \dots, g_n))$ for all elements $\sigma = (f_1, \dots, f_n)$ and $\tau = (g_1, \dots, g_n)$ in \mathcal{E} . The monoid \mathcal{E} is anti-isomorphic to $\text{End}_{\mathbb{K}}\mathbb{K}[x_1, \dots, x_n]$ (the monoid of polynomial endomorphisms).

We denote by $\mathcal{G} = \text{GA}_n(\mathbb{K})$ the group of invertible elements of \mathcal{E} . Note that \mathcal{G} is anti-isomorphic to the group $\text{Aut}_{\mathbb{K}}\mathbb{K}[x_1, \dots, x_n]$ (the group of polynomial automorphisms) and isomorphic to the group of polynomial automorphisms of the affine n -space $\mathbb{A}_{\mathbb{K}}^n$.

We define the degree of $\sigma = (f_1, \dots, f_n) \in \mathcal{E}$ by $\deg(\sigma) = \max_{1 \leq i \leq n} \{\deg(f_i)\}$. We denote by $\mathcal{A} = \text{Aff}_n(\mathbb{K}) = \{\sigma \in \mathcal{G}; \deg(\sigma) = 1\}$ the *affine subgroup*, by $\mathcal{B} = \text{BA}_n(\mathbb{K}) = \{(\alpha_1 x_1 + p_1, \dots, \alpha_n x_n + p_n); \alpha_i \in \mathbb{K}^*, p_i \in k[x_{i+1}, \dots, x_n], \forall i\}$ the *triangular subgroup* and by $\mathcal{T} = \text{TA}_n(\mathbb{K})$ the *tame subgroup* i. e. the subgroup generated by \mathcal{A} and \mathcal{B} (see [5] for more informations about polynomial automorphisms).

Given a positive integer $d \in \mathbb{N}_+$, the set $\mathcal{E}_{\leq d}$ of polynomial endomorphisms of degree $\leq d$ is a \mathbb{K} -vector space of dimension $N = \binom{n+d}{d}$ (the degree of a polynomial endomorphism is the maximum of the degree of its components) and we can consider $\mathcal{E}_{\leq d}$ as an algebraic variety by transfer of the structure of the affine N -space $\mathbb{A}_{\mathbb{K}}^N$. We consider the Zariski topology on $\mathcal{E}_{\leq d}$ associated with this structure.

Following Shafarevich (see [12]), since for all positive integers $d \in \mathbb{N}_+$, $\mathcal{E}_{\leq d}$ is closed in $\mathcal{E}_{\leq d+1}$, we can endow $\mathcal{E} = \bigcup_{d \geq 1} \mathcal{E}_{\leq d}$ with the inductive limit topology. A subset $A \subset \mathcal{E}$ is closed in \mathcal{E} for this topology if and only if $A \cap \mathcal{E}_{\leq d}$ is closed in

$\mathcal{E}_{\leq d}$ for all $d \in \mathbb{N}_+$. We consider the restriction of this inductive limit topology on \mathcal{G} and we set $\mathcal{G}_{\leq d} = \mathcal{G} \cap \mathcal{E}_{\leq d}$.

In this paper, we are interested with the question:

Question 1.1 *Which are the closed subgroups of \mathcal{G} containing \mathcal{A} ?*

We prove the following result in the characteristic zero case:

Theorem 1.2 ($n \geq 2$, $\text{char}(\mathbb{K}) = 0$) *If \mathcal{H} is a closed subgroup of \mathcal{G} strictly containing \mathcal{A} then $\mathcal{T} \subset \mathcal{H}$.*

In [2] (see Theorem 2.10), Bodnarchuk claimed to prove this result with even the stronger conclusion $\mathcal{H} = \mathcal{G}$. But his proof is based on a erroneous result of Shafarevich (cited as Theorem 2.7 in [2]). Recently, Furter and Kraft (see [7]) proved that, in the case $n = 3$ and $\mathbb{K} = \mathbb{C}$, the subgroup \mathcal{T}' is closed in \mathcal{G}' where \mathcal{G}' is the subgroup of \mathcal{G} fixing x_3 and $\mathcal{T}' = \mathcal{G}' \cap \mathcal{T}$. This result is based on the Shestakov-Umirbaev theory (see [13]) which implies that $\mathcal{T}' \neq \mathcal{G}'$. With notations of Theorem 2.7 in [2], taking $G = \mathcal{G}'$, $H = \mathcal{T}'$ and $f : \mathcal{T}' \rightarrow \mathcal{G}'$ the canonical inclusion, $(df)_e$ is the canonical isomorphism between the tangent spaces $T_{e,H}$ and $T_{e,G}$, but f is not an isomorphism because of the Shestakov-Umirbaev theorem.

In the case $n = 2$, the Jung-van der Kulk theorem (see [8, 9]) says that $\mathcal{T} = \mathcal{G}$ and Theorem 1.2 implies that the affine group is a maximal closed subgroup of the plane polynomial automorphism group. This gives a positive answer to a question of Furter (see Question 1 and Question 1.11 in [6]).

In the case $n = 3$, we know that $\mathcal{T} \neq \mathcal{G}$. Recently, Poloni and the author (see [4]) proved that some explicit families of automorphisms in $\mathcal{G} \setminus \mathcal{T}$ are in the closure of \mathcal{T} in \mathcal{G} (see also [10]). But we don't know if \mathcal{T} is dense in \mathcal{G} or not. In particular, we don't know if the Nagata automorphism (see [11]) is in the closure of \mathcal{T} in \mathcal{G} .

The situation in positive characteristic is more complex and will be studied in an other paper.

2 Closed subgroup

In this section, we use notations introduced in the last section. We do not assume that \mathbb{K} has characteristic zero. We consider the following two sets of variables $\mathbf{x} = \{x_1, \dots, x_n\}$ and $\hat{\mathbf{x}} = \{x_2, \dots, x_n\}$. We denote by I (resp. \hat{I}) the ideal of $\mathbb{K}[\mathbf{x}] = \mathbb{K}[x_1, \dots, x_n]$ (resp. $\mathbb{K}[\hat{\mathbf{x}}] = \mathbb{K}[x_2, \dots, x_n]$) generated by \mathbf{x} (resp. $\hat{\mathbf{x}}$). We say that an automorphism $\phi = (f_1, \dots, f_n) \in \mathcal{G}$ has a *affine part equal to id* if $f_i \equiv x_i$ modulo I^2 for all $i \in \{1, \dots, n\}$. We prove the following result which is the main theorem of this paper.

Theorem 2.1 ($n \geq 2$) *Let $\phi = (f_1, \dots, f_n) \in \mathcal{G} \setminus \mathcal{A}$ be an automorphism of degree $d \geq 2$. We assume that ϕ has a affine part equal to id and $f_1 \neq x_1$.*

a) The polynomial $g_0(\hat{\mathbf{x}}) := f_1(0, \hat{\mathbf{x}})$ is not zero and has a \hat{I} -adic valuation $w \geq 2$.

b) Let $\mathcal{H} := \langle \mathcal{A}, \phi \rangle$ be the subgroup of \mathcal{G} generated by $\mathcal{A} \cup \{\phi\}$. The closure of $\mathcal{H} \cap \mathcal{G}_{\leq d}$ in $\mathcal{G}_{\leq d}$ contains the automorphism $(x_1 + h(\hat{\mathbf{x}}), \hat{\mathbf{x}})$ where $h(\hat{\mathbf{x}})$ is the homogeneous part of $g_0(\hat{\mathbf{x}})$ of degree w .

Proof We write $f_1 = \sum_{k=0}^m g_k x_1^k$ where $m = \deg_{x_1} f_1 \geq 1$ and $g_i \in \mathbb{K}[\hat{\mathbf{x}}]$.

a) By contradiction, we assume that $g_0 = 0$. Then x_1 divides f_1 . Since f_1 is a coordinate, f_1 is an irreducible polynomial and using that $f_1 \equiv x_1$ modulo I^2 , we deduce $f_1 = x_1$. Impossible. Since the affine part of ϕ is id , it's clear that $2 \leq w < +\infty$.

b) We consider the torus action $\alpha_t := (t^w x_1, t\hat{\mathbf{x}}) \in \text{Aff}_n(\mathbb{K}(t))$ and we compute

$$\phi_t := \alpha_t^{-1} \phi \alpha_t = (t^{-w} f_1(t^w x_1, t\hat{\mathbf{x}}), t^{-1} f_2(t^w x_1, t\hat{\mathbf{x}}), \dots, t^{-1} f_n(t^w x_1, t\hat{\mathbf{x}})).$$

Using that the affine part of ϕ is id , we have $g_1(0) = 1$ and

$$t^{-w} f_1(t^w x_1, t\hat{\mathbf{x}}) = t^{-w} \sum_{k=0}^m g_k(t\hat{\mathbf{x}}) t^{kw} x_1^k \equiv t^{-w} g_0(t\hat{\mathbf{x}}) + x_1 \equiv x_1 + h(\hat{\mathbf{x}}) \pmod{t\mathbb{K}[\mathbf{x}, t]}$$

and $t^{-1} f_i(t^w x_1, t\hat{\mathbf{x}}) \equiv x_i \pmod{t\mathbb{K}[\mathbf{x}, t]}$, for all $i \in \{2, \dots, n\}$. Using that $\text{Jac}(\phi_t) = \text{Jac}(\phi) \in \mathbb{K}^*$, we deduce from the overring principle (see Lemma 1.1.8 p. 5 in [5]) that $\phi_t \in \text{GA}_n(\mathbb{K}[t])$. For all $t_0 \in \mathbb{K}^*$, the automorphism $\phi_{t \rightarrow t_0}$ is in $\mathcal{H} \cap \mathcal{G}_{\leq d}$ and we deduce that $(x_1 + h(\hat{\mathbf{x}}), \hat{\mathbf{x}}) = (x_1 + h(\hat{\mathbf{x}}), x_2, \dots, x_n) = \phi_{t \rightarrow 0}$ is in the closure of $\mathcal{H} \cap \mathcal{G}_{\leq d}$ in $\mathcal{G}_{\leq d}$.

Remark: This proof is based on a classical technique which consists to conjugate an automorphism by an action of the torus. See the proof of Theorem 4.6 in [4] or the proof of Lemma 4.1 in [1] for other situations where this technique is used.

Corollary 2.2 *Let $\phi \in \mathcal{G} \setminus \mathcal{A}$ be an automorphism of degree $d \geq 2$. Let $\mathcal{H} := \langle \mathcal{A}, \phi \rangle$ be the subgroup of \mathcal{G} generated by $\mathcal{A} \cup \{\phi\}$. Then the closure of $\mathcal{H} \cap \mathcal{G}_{\leq d}$ in $\mathcal{G}_{\leq d}$ contains an element of $\mathcal{B} \setminus \mathcal{A}$.*

Proof Changing ϕ to $\alpha^{-1} \phi$ where $\alpha \in \mathcal{A}$ is the affine part of ϕ , we can assume that the affine part of ϕ is id . Since $\phi = (f_1, \dots, f_n)$ is not affine, there exists $i \in \{1, \dots, n\}$ such that $f_i \neq x_i$. Changing ϕ to $\sigma \phi \sigma$ where $\sigma \in \mathfrak{S}_n \subset \mathcal{A}$ is the transposition $x_1 \leftrightarrow x_i$, we can assume that $f_1 \neq x_1$ and apply Theorem 2.1.

3 Characteristic zero

In this section, we assume that \mathbb{K} has characteristic zero. We recall two complementary results. Theorem 3.1, in the case $n = 2$ (we recall that when

$n = 2$, we have $\mathcal{T} = \mathcal{G}$), is due to Furter (see Theorem D in [6]). Theorem 3.2, in the case $n \geq 3$, is due to Bodnarchuk (see [3]).

Theorem 3.1 ($n = 2$, $\text{char}(\mathbb{K}) = 0$) *For all $\beta \in \mathcal{B} \setminus \mathcal{A}$ we have $\overline{\langle \mathcal{A}, \beta \rangle} = \mathcal{G}$.*

Theorem 3.2 ($n \geq 3$, $\text{char}(\mathbb{K}) = 0$) *For all $\beta \in \mathcal{B} \setminus \mathcal{A}$ we have $\langle \mathcal{A}, \beta \rangle = \mathcal{T}$.*

Using this two results, Corollary 2.2 implies Theorem 1.2.

Theorem 1.2 ($n \geq 2$, $\text{char}(\mathbb{K}) = 0$). *If \mathcal{H} is a closed subgroup of \mathcal{G} strictly containing \mathcal{A} then $\mathcal{T} \subset \mathcal{H}$.*

Proof Since \mathcal{H} strictly containing \mathcal{A} , there exists $\phi \in \mathcal{H} \setminus \mathcal{A}$ of degree $d \geq 2$. By Corollary 2.2, the closure of $\langle \mathcal{A}, \phi \rangle$ in $\mathcal{G}_{\leq d}$ contains an element $\beta \in \mathcal{B} \setminus \mathcal{A}$. Using that \mathcal{H} is closed in \mathcal{G} we deduce $\beta \in \overline{\langle \mathcal{A}, \phi \rangle} \subset \overline{\mathcal{H}} = \mathcal{H}$.

a) If $n = 2$, by Theorem 3.1, we have $\mathcal{T} = \mathcal{G} = \overline{\langle \mathcal{A}, \beta \rangle} \subset \overline{\mathcal{H}} = \mathcal{H}$.

b) If $n \geq 3$, by Theorem 3.2, we have $\mathcal{T} = \langle \mathcal{A}, \beta \rangle \subset \mathcal{H}$.

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